Energy of Superlinear Preferential Attachment Trees

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Abstract

Preferential attachment (PA) trees model growing networks by linking a new vertex to an existing one with probability proportional to its degree. We study the energy for superlinear PA trees, where attachment probabilities scale as d^α with $\alpha > 1$. We prove that superlinear PA trees are asymptotically hypoenergetic almost surely, with $\limsup_{n\to\infty} \mathcal{E}(T_n)/n = 0$. Moreover, we fully specify the different growth regimes of energy for $\alpha > 1$.

Contents

1 Introduction **1**
1.1 Definitions 1
1.2 Motivation 2

2 Main results **3**

3 Preliminary lemmas **3**
3.1 Minimum and maximum energy trees 4
3.2 Upper bound using degrees of vertices 4
3.3 Limiting distributions of degrees in superlinear PAs 4

4 Main proofs **5**
4.1 Proof of Theorem 2.1 (a) 5
4.2 Proof of Theorem 2.1 (b) 7
4.3 Proof of Theorem 2.1 (c) 8

5 Conclusion **10**

1 Introduction

1.1 Definitions

Let G be a graph with n vertices, and let $A(G)$ be its adjacency matrix. The eigenvalues of $A(G)$ are denoted as $\lambda_1, \lambda_2, \ldots, \lambda_n$. The *energy* of the graph G, denoted by $\mathcal{E}(G)$, is defined as the sum of the absolute values of the eigenvalues of $A(G)$:

$$\mathcal{E}(G) = \sum_{j=1}^{n} |\lambda_j|.$$

Preferential attachment (PA) trees are a fundamental class of random tree models used to describe growing networks, capturing the "rich get richer" phenomenon. These models are constructed recursively: a new vertex is added one at a time and linked to an existing vertex v with a probability proportional to a function f of its degree. More exactly the probability to attach to v would be $f(deg(v))/\sum_{w\in V} f(deg(w))$, where V denotes the vertex set.

A PA tree is thus governed by an *attachment kernel* $f(k)$, which determines the likelihood of a vertex with degree k receiving a new edge. This function defines some main regimes:

- **Linear regime**: $f(k) = k$, leading to power-law degree distribution [Bollobás et al., 2001]
- **Superlinear regime**: $f(k) = k^\alpha$ with $\alpha > 1$, where a dominant vertex emerges [Oliveira and Spencer, 2005]
- **Sublinear regime**: $f(k) = k^\alpha$ with $0 < \alpha < 1$, where the degree distribution follows a stretched exponential [Dereich and Mörters, 2009]

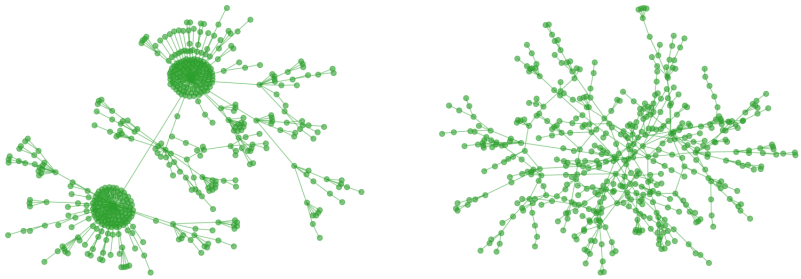


Figure 1: Superlinear PA (left) and sublinear PA (right)

1.2 Motivation

The study in [Arizmendi and Dominguez, 2022] investigates the energy of PA trees in the linear regime, showing that with probability tending to 1, as the size tends to infinity, the trees are *hypoenergetic*, meaning their energy is, with high probability, smaller than their size. The concluding remarks of this paper suggest exploring how energy changes across different PA regimes, particularly with respect to the exponent α in the attachment kernel. At the end of [Arizmendi and Dominguez, 2022], the authors claim:

Conjecture 5.1 Let T_n be a PA tree with parameter α:

1. For every α there exists $g(\alpha)$ such that the following limit exists almost surely
$$\lim_{n\to\infty} \frac{\mathcal{E}(T_n)}{n} = g(\alpha)$$
2. For $\alpha < .79$, $g(\alpha) > 1$. In particular, taking $\alpha = 0$, this would imply that random recursive random trees are not hypoenergetic
3. For $\alpha > .81$, $g(\alpha) < 1$
4. $g(\alpha)$ is strictly decreasing on α and continuous
5. $g(\alpha) \to 0$ as $\alpha \to \infty$ and $g(\alpha) \to 1.273$ as $\alpha \to 0$

The primary objective of this article is to fully characterize the growth regimes of graph energy $\mathcal{E}(T_n)$ for the superlinear PA trees. We show that for superlinear PA trees, $\mathcal{E}(T_n)/n \to 0$ a.s. as $n \to \infty$, or, with the notation above, that $g(\alpha) = 0$ for $\alpha > 1$. We actually establish a more precise result which identifies the order of magnitude of $\mathcal{E}(T_n)$ depending on the value of $\alpha > 1$.

2 Main results

Theorem 2.1. *Assume that T_n is a superlinear PA, i.e. $\alpha > 1$.*

(a) When $1 < \alpha < 1.5$, we have:
$$0 < \frac{1}{2-\alpha} \leq \liminf_{n\to\infty} \frac{\mathcal{E}(T_n)}{n^{2-\alpha}} \leq \limsup_{n\to\infty} \frac{\mathcal{E}(T_n)}{n^{2-\alpha}} \leq \frac{2}{2-\alpha} \quad a.s.$$

(b) When $1.5 \leq \alpha \leq 2$, $\exists C_\alpha$ constant such that:
$$2 \leq \liminf_{n\to\infty} \frac{\mathcal{E}(T_n)}{\sqrt{n}} \leq \limsup_{n\to\infty} \frac{\mathcal{E}(T_n)}{\sqrt{n}} \leq C_\alpha \quad a.s.$$

(c) In addition, when $\alpha > 2$ we have:
$$\mathcal{E}(T_n) \sim 2\sqrt{n} \quad \text{as } n \to \infty \quad a.s.$$

3 Preliminary lemmas

In this section, we present key lemmas, recalled from previous works, that will serve as fundamental tools in our main proofs.

3.1 Minimum and maximum energy trees

A well-known result from graph energy theory [Li et al., 2012] states that the energy of a star graph is given by:

$$\mathcal{E}(S_n) = 2\sqrt{n-1},$$

while the energy of a path graph is approximately:

$$\mathcal{E}(P_n) \approx 1.273n.$$

In particular, these are the trees with minimum, respectively maximum energy.

Lemma 3.1 (Li et al., 2012, p. 34)**.** *For any tree T_n on n vertices, we have:*

$$\mathcal{E}(S_n) \leq \mathcal{E}(T_n) \leq \mathcal{E}(P_n)$$

3.2 Upper bound using degrees of vertices

We now state a fundamental inequality from [Arizmendi and Dominguez, 2022], which provides an upper bound on the energy of a tree in terms of its vertex degrees.

Lemma 3.2 (Arizmendi and Dominguez, 2022, p. 5)**.** *Let T be a tree with degrees $\Delta = d_1 \geq \cdots \geq d_n$, $n \geq 3$. Then*

$$\mathcal{E}(T) \leq \sum_{i=2}^{n} 2\sqrt{d_i - 1} + 2\sqrt{\Delta} \leq \sum_{i=1}^{n} 2\sqrt{d_i - 1} + 1$$

3.3 Limiting distributions of degrees in superlinear PAs

Denote by:

- $Z_k(n)$ the number of vertices with degree k at time n
- $\Phi_k(n) = \sum_{\ell \geq k} Z_\ell(n)$ the count of vertices with degree at least k at time n
- $V(n) = \max\{k : Z_k(n) \geq 1\}$ the largest degree among the vertices at time n

We recall important results, all from the [Sethuraman and Venkataramani, 2019] paper, which prove the limiting distribution of $Z_k(n)$, $\Phi_k(n)$ and $V(n)$ in a.s. limit for the case of superlinear PAs.

Lemma 3.3 (Sethuraman and Venkataramani, 2019, p. 3)**.** *For a superlinear PA with attachment kernel $f(k) = k^\alpha$ with $\alpha > 1$, the following structure for the degree sequence holds:*

(a) When $\frac{\alpha}{\alpha-1} > j \geq 1$*, a.s. we have:*

$$\lim_{n\to\infty} \frac{1}{n^{j-(j-1)\alpha}} Z_j(n) = a_j \quad \textit{where} \quad a_j = \prod_{i=2}^{j} \frac{(i-1)^\alpha}{i-(i-1)\alpha}$$

(b) When $j = \frac{\alpha}{\alpha-1} \geq 2$ *is an integer, a.s. we have:*

$$\lim_{n\to\infty} \frac{1}{\log(n)} Z_j(n) = b_j \quad \textit{where} \quad b_j = w(j-1)a_{j-1}$$

(c) When $j > \frac{\alpha}{\alpha-1}$*,*

$$\lim_{n\to\infty} \Phi_j(n) < \infty \quad \textit{a.s.}$$

(d) Also, a.s., for $n \geq n_0$*, where* n_0 *is a random time,* $V(n)$ *is achieved at a fixed (random) vertex, and*

$$\lim_{n\to\infty} \frac{1}{n} V(n) = 1.$$

4 Main proofs

All proofs will use Lemma 3.2, in a slightly modified but equivalent form:

$$\mathcal{E}(T_n) \leq 2\sum_{j=1}^{n} Z_j(n)\sqrt{j-1} + 1 \tag{1}$$

In particular, note that we can ignore vertices of degree 1 in the sum, because they don't contribute to the right-hand side of the inequality.
Denote by k the smallest integer strictly greater than $\alpha/(\alpha-1)$. Note in particular $k \geq 2$. Then we can separate the above inequality the following way:

$$\begin{aligned} \mathcal{E}(T_n) &\leq \sum_{j=2}^{k-1} Z_j(n) 2\sqrt{j-1} + \sum_{j=k}^{n} Z_j(n) 2\sqrt{j-1} + 1 \\ &\leq \sum_{j=2}^{k-1} Z_j(n) 2\sqrt{k-1} + \Phi_k(n) 2\sqrt{n-1} + 1 \end{aligned} \tag{2}$$

4.1 Proof of Theorem 2.1 (a)

Assume now $1 < \alpha < 1.5$. Note in this case $k \geq 3$. Trying to refine the upper bound (2), we have:

$$\frac{\mathcal{E}(T_n)}{n^{2-\alpha}} \leq \frac{2Z_2(n)}{n^{2-\alpha}} + \sum_{j=3}^{k-1} \frac{Z_j(n)}{n^{2-\alpha}} 2\sqrt{k-1} + \frac{\Phi_k(n)}{n^{2-\alpha}} 2\sqrt{n-1} + \frac{1}{n^{2-\alpha}} \tag{3}$$

Because of Lemma 3.3 (a), $Z_j(n)$ grows asymptotically like $n^{j-(j-1)\alpha}$ for $j \leq k-2$, and then, using the fact that $j-(j-1)\alpha < 2-\alpha$ for $j \geq 3$:

$$\lim_{n\to\infty} \frac{Z_j(n)}{n^{2-\alpha}} = 0 \ \ a.s. \ \text{ for } \ 3 \leq j \leq k-2 \tag{4}$$

If $\alpha/(\alpha-1)$ is not an integer, Lemma 3.3 (a) applies to $Z_{k-1}(n)$ as well, like above. If $\alpha/(\alpha-1)$ is an integer, it would be equal to $k-1$ (by definition of k). Then, by Lemma 3.3 (b), $Z_{k-1}(n)$ grows asymptotically like $\log(n)$. Either way, $\lim_{n\to\infty} Z_{k-1}(n)/n^{2-\alpha} = 0 \ \ a.s.$

Hence, by finite intersection of a.s. events, the sum on the right-hand side of (3) goes to zero a.s. as $n \to \infty$. Because $2-\alpha > 0.5$ and $\limsup_{n\to\infty} \Phi_k(n)$ is finite a.s by Lemma 3.3 (c), the third term on the right-hand side of (3):

$$\lim_{n\to\infty} \frac{\Phi_k(n)}{n^{2-\alpha}} 2\sqrt{n-1} = 0 \ \ a.s. \tag{5}$$

Thus, the only non-zero term on the right-hand side of (3) as we take the limsup as $n \to \infty$ will be:

$$\lim_{n\to\infty} \frac{2Z_2(n)}{n^{2-\alpha}} \overset{\text{Lemma 3.3}}{=} \frac{2}{2-\alpha} \tag{6}$$

So the right inequality of Theorem 2.1 (a) is satisfied.

Now for the lower bound, we will use the following result from [Arizmendi and Arizmendi, 2020]: if G has a matching of size k, then $\mathcal{E}(G) \geq 2k$. We will thus try to lower bound the maximal matching of T_n.

We know from an inductive argument, that given a rooted tree containing more than one vertex, there is a matching which covers all non-leaf nodes. In our case, we can choose the root to be any vertex of degree more than 2 to establish a hierarchy. We match the root to an arbitrary child, and recurse on the remaining trees obtained by removing the root and its chosen child. Each remaining tree, is either an isolated vertex, if the new root vertex was initially a leaf, or itself a rooted tree containing more than one vertex, if the new root vertex was a non-leaf in the original tree, and so the induction hypothesis applies.

In particular, all vertices of degree 2 are internal nodes and hence, we can find a matching which covers all of them, so of size greater or equal to $Z_2(n)/2$. Hence, $\mathcal{E}(T_n) \geq Z_2(n)$. So, using (6), $\liminf_{n\to\infty} \mathcal{E}(T_n)/n^{2-\alpha} \geq 1/(2-\alpha) \ \ a.s.$. Thus the left inequality of Theorem 2.1 (a) is satisfied.

Remark 4.1. *Notice that in the figure below the energy growth is bounded when divided by $n^{2-\alpha}$, which aligns with our theoretical result.*

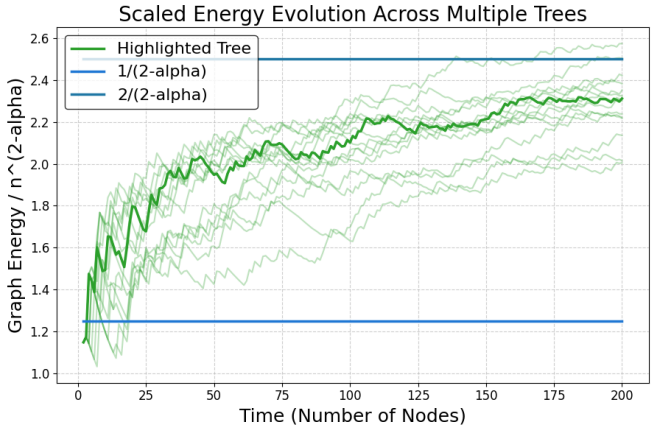


Figure 2: Energy/$n^{2-\alpha}$ evolution for $\alpha = 1.2$

4.2 Proof of Theorem 2.1 (b)

Assume now $1.5 \leq \alpha \leq 2$. Trying to refine the upper bound (2), we have:

$$\frac{\mathcal{E}(T_n)}{\sqrt{n}} \leq \frac{2Z_2(n)}{\sqrt{n}} + \sum_{j=3}^{k-1} \frac{Z_j(n)}{\sqrt{n}} 2\sqrt{k-1} + \frac{\Phi_k(n)}{\sqrt{n}} 2\sqrt{n-1} + \frac{1}{\sqrt{n}} \tag{7}$$

Because of Lemma 3.3 (a), $Z_j(n)$ grows asymptotically like $n^{j-(j-1)\alpha}$ for $j \leq k-2$, and using the fact that $j-(j-1)\alpha < 2-\alpha \leq 0.5$ for $j \geq 3$:

$$\lim_{n\to\infty} \frac{Z_j(n)}{\sqrt{n}} = 0 \ \ a.s. \ \text{ for } \ 3 \leq j \leq k-2 \tag{8}$$

Again, if $\alpha/(\alpha-1)$ is not an integer, Lemma 3.3 (a) applies to $Z_{k-1}(n)$ as well, like above. If $\alpha/(\alpha-1)$ is an integer, it would be equal to $k-1$ (by definition of k). Then, by Lemma 3.3 (b), $Z_{k-1}(n)$ actually grows asymptotically like $\log(n)$. Either way, $\lim_{n\to\infty} Z_{k-1}(n)/\sqrt{n} = 0 \ \ a.s.$
Taking the limsup in (7), we get by a finite intersection of a.s. events that:

$$\limsup_{n\to\infty} \frac{\mathcal{E}(T_n)}{\sqrt{n}} \leq 2\left(\limsup_{n\to\infty} \frac{Z_2(n)}{\sqrt{n}} + \limsup_{n\to\infty} \Phi_k(n)\right) \ \ a.s. \tag{9}$$

By Lemma 3.3, $\limsup_{n\to\infty} Z_2(n)/\sqrt{n}$ exists and is finite a.s. More exactly:

- If $\alpha = 1.5$, then the limit is given by $\lim_{n\to\infty} \frac{Z_2(n)}{\sqrt{n}} = \frac{1}{2-\alpha} = 2$
- If $\alpha > 1.5$, then the limit is zero:

– For $1.5 < \alpha < 2$, $Z_2(n)$ grows like $n^{2-\alpha}$, which is asymptotically smaller than $\sqrt{n}$

– For $\alpha = 2$, $Z_2(n)$ grows like $\log n$ by Lemma 3.3(b), which is still asymptotically smaller than $\sqrt{n}$

Moreover, $\limsup_{n\to\infty} \Phi_k(n)$ is finite a.s by Lemma 3.3 (c). So, the right inequality in Theorem 2.1 (b) is proved, denoting $C_\alpha = 2(\limsup_{n\to\infty} Z_2(n)/\sqrt{n} + \limsup_{n\to\infty} \Phi_k(n))$, which is finite a.s.

The left inequality in Theorem 2.1 (b) comes from Lemma 3.1, which lower bounds the energy of a tree using the energy of a star:

$$\mathcal{E}(T_n) \geq 2\sqrt{n-1} \tag{10}$$

Remark 4.2. *Notice that in the Figure 3 the energy growth plateaus when divided by $\sqrt{n}$, which aligns with our theoretical result.*

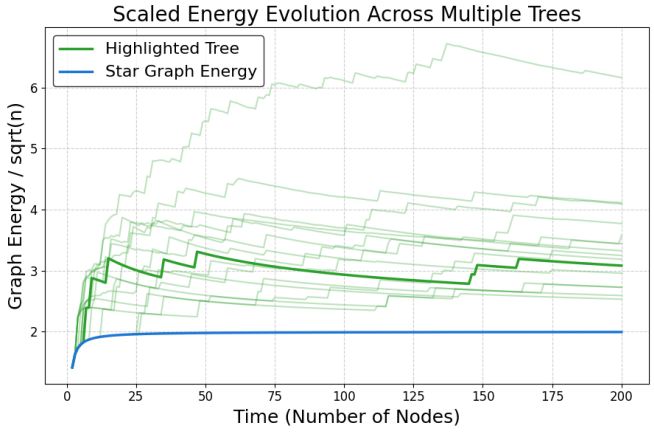


Figure 3: Energy/$\sqrt{n}$ evolution for $\alpha = 1.8$

4.3 Proof of Theorem 2.1 (c)

For this theorem we have to improve the upper bound obtained in (2). In particular, because we now consider $\alpha > 2$, we note that our k (which we recall to be the smallest integer strictly greater than $\frac{\alpha}{\alpha-1}$) will be $k = 2$. Thus Lemma 3.3 (c) holds for $j = 2$:

$$\lim_{n\to\infty} \Phi_2(n) < \infty \ \ a.s. \tag{11}$$

Moreover, specifying Lemma 3.3 (a) for $j = 1$, we get:

$$\lim_{n\to\infty} \frac{Z_1(n)}{n} = 1 \ \ a.s. \tag{12}$$

and recalling Lemma 3.3 (d), we get:

$$\lim_{n\to\infty} \frac{V(n)}{n} = 1 \ \ a.s. \tag{13}$$

where $V(n) = \max\{k : Z_k(n) \geq 1\}$ denotes the largest degree among the vertices in T_n. The paper also shows that this largest degree is achieved at a fixed random vertex and thus in particular $Z_{V(n)}(n) = 1$ a.s. for n sufficiently large. Using this notation, we also get from the maximality of $V(n)$ that:

$$Z_j(n) = 0 \ \text{ for } \ j > V(n) \tag{14}$$

In particular, the above tell us that:

$$\text{a.s. } \exists N_\varepsilon \ \text{ s.t. for } \ n \geq N_\varepsilon, \ \begin{cases} Z_1(n) > (1-\varepsilon)n \\ V(n) > (1-\varepsilon)n \\ Z_{V(n)}(n) = 1 \end{cases} \tag{15}$$

Since vertex degrees add up to 2 times the number of edges, we have:

$$\sum_{j=1}^{n} Z_j(n) j = 2(n-1) \tag{16}$$

Combining the previous remarks (14), (15) and (16) together, we get that a.s. $\exists N_\varepsilon$ s.t for $n \geq N_\varepsilon$:

$$\sum_{j=2}^{V(n)-1} Z_j(n) j = 2(n-1) - Z_1(n) - V(n) < 2\varepsilon n \tag{17}$$

We will use inequality (1) again, but refine it:

$$\begin{aligned} \mathcal{E}(T_n) &\leq \sum_{j=2}^{n} Z_j(n) 2\sqrt{j-1} + 1 \overset{(14)}{=} \sum_{j=2}^{V(n)} Z_j(n) 2\sqrt{j-1} + 1 \\ &= \sum_{j=2}^{V(n)-1} Z_j(n) 2\sqrt{j-1} + 2\sqrt{V(n)-1} + 1 \end{aligned} \tag{18}$$

Then we can apply a Cauchy-Schwartz inequality, and we get that a.s. $\exists N_\varepsilon$ s.t for $n \geq N_\varepsilon$:

$$\begin{aligned} \sum_{j=2}^{V(n)-1} Z_j(n)\sqrt{j-1} &\leq \sqrt{\left(\sum_{j=2}^{V(n)-1} Z_j(n)\right)\left(\sum_{j=2}^{V(n)-1} Z_j(n)(j-1)\right)} \\ &\overset{(17)}{\leq} \sqrt{\Phi_2(n)}\sqrt{2\varepsilon n} = \sqrt{2\varepsilon \Phi_2(n) n} \end{aligned} \tag{19}$$

Using (19) and the fact that $V(n) \leq n$ (as there is no vertex of degree larger than n at time n) in inequality (18), we deduce that a.s. $\exists N_\varepsilon$ s.t for $n \geq N_\varepsilon$:

$$\frac{\mathcal{E}(T_n)}{\sqrt{n}} \leq 2\sqrt{2\varepsilon\Phi_2(n)} + 2 + \frac{1}{\sqrt{n}} \tag{20}$$

Thus, taking the limsup above, we have that a.s.:

$$\limsup_{n\to\infty} \frac{\mathcal{E}(T_n)}{\sqrt{n}} \leq 2\sqrt{2\varepsilon \limsup_{n\to\infty} \Phi_2(n)} + 2 \tag{21}$$

But in particular, $\limsup_{n\to\infty} \Phi_2(n)$ is a.s. finite by (11), and the above holds a.s. for every $\varepsilon > 0$. Thus, letting $\varepsilon \to 0$, we have that a.s.:

$$\limsup_{n\to\infty} \frac{\mathcal{E}(T_n)}{\sqrt{n}} \leq 2 \tag{22}$$

Combining the last inequality with (10), we can conclude that indeed a.s.:

$$\lim_{n\to\infty} \frac{\mathcal{E}(T_n)}{\sqrt{n}} = 2 \tag{23}$$

Remark 4.3. *Below we present some simulations to support the proven theorem:*

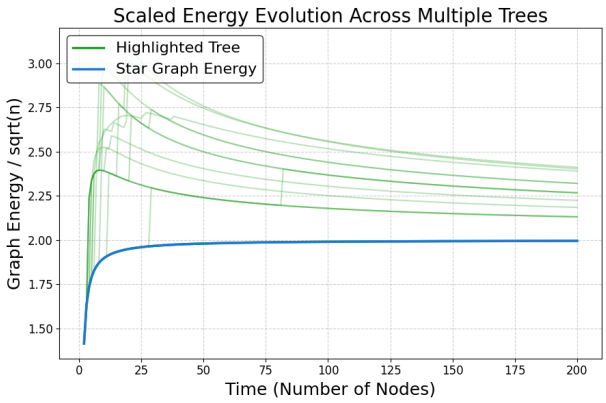


Figure 4: Energy/$\sqrt{n}$ for $\alpha = 2.6$

5 Conclusion

In this article, we proved that almost surely superlinear PA trees are asymptotically hypoenergetic, with energy growing sublinearly in the number of vertices. Additionally, we fully specified the asymptotic behaviour for the different growth regimes in the case $\alpha > 1$.

Beyond the theoretical results, we have also developed a GitHub package for simulating preferential attachment trees across different regimes. This tool facilitates further experimental verification and analysis, and it is what we used to generate the plots.

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